Connecting Mathematical Ideas:
Middle School Video Cases to Support Teaching and Learning

Ch. 1: Opening the Door to My Classroom

It was disheartening to read about Cathy’s confidence in her teaching methods only to come to the sudden realization that they were not effective. I could imagine the frustration she faced when her students had forgotten materials she presented just months earlier. I strongly agreed with the statement that “many students [who have] master[ed] procedures without mastering the underlying substance” (8). Many students fail to see the connections among mathematical ideas and purely focus on what it is that they need to do rather than why—as I have witnessed this through my work with students and have experienced this feeling myself as a past math student. I hope to carry the same philosophy as Cathy into my future classroom: “The act of building connections and relationships is at the heart of mathematical proficiency” (11).

It was interesting to read about the personal element of teaching that Cathy felt and how difficult it was to watch her taped lessons. I admire that she was able to share her work with others in order to help pre-service teachers like myself learn more about teaching. I also appreciate that the video clips are not entirely of ideal lessons, but instead the “interesting, unexpected, and sometimes difficult moments that occurred during ordinary lessons” (10). This made me think back to the initial discomfort I felt when I had to tape myself in past Curriculum and Instruction courses but I plan to do so in the future. I own a FlipCam and hope that I can use it to record my teaching for later reflection and refinement.

I thought Cathy’s beliefs were very wise and I feel that these are views I want to voice to my students in my future classroom. I might even include these things on classroom posters/course syllabi. Cathy’s core beliefs (in short) are:
• Learning mathematics means making sense of mathematical relationships
• Teaching mathematics means helping all students learn to think mathematically
• Mathematics extends beyond arithmetic
• Mistakes and confusion are an essential part of understanding
• There are many ways to be good at math
• There are multiple ways to approach problems; even ones with only one answer
• Mathematics should make sense
• Talking to peers about mathematical ideas helps others understand in different ways
• You understand what you can explain

**Ch. 2: Building on Student Ideas, The Border Problem, Part 1**

This chapter centers on the notion that “emphasis on pattern generalization supports the development of algebraic thinking and that the concept of function is fundamental to the ideas of algebra” (13). She suggests staying away from encouraging students to find an algebraic rule and putting more emphasis on recognizing that different visualizations of a pattern can be described symbolically in equivalent algebraic expressions. The skill of generalization seems to be a missing component amongst middle and high school mathematics students. It is important to highlight that not every student will interpret certain concepts in the same way and that mathematics is a flexible subject that allows students to form their personalized method of understanding. “A teacher who presents a subject matter in all of its complexity makes it more accessible by opening a multiplicity of paths into it” (14).

Cathy’s lesson involved investigating growth patterns embedded in discrete polynomial functions using ‘The Border Problem’. This problem requires students to calculate the number of colored squares in the border of an n-by-n grid visualli—without writing anything down and without counting one-by-one. This was a clever task implementation because Cathy was aware of the results her strategy would produce. This showed me that students don’t always need a pencil and paper in front of them to do math. This also highlighted that whenever giving
students directions for a task you would like them to complete, it is important to be explicit in the behavior you expect from them. Following this, she not only had students share their ideas but come to the front of the board and visually show the class what their method was. This is something I definitely want to implement in my classroom because I feel that it will be valuable not only for the student at the front of the room to verbally explain their mindset, but also for other students to practice learning from their peers and to hear different perspectives other than their own. The fact that she emphasized there were multiple methods at arriving at the same answer stressed the freedom granted to students in their mathematical discoveries. I also really liked how Cathy represented student ideas in a variety of ways: she allowed them to verbally explain, visually show their train of thought using the 10-by-10 grid projected at the front of the room, and represent their thinking using numerical representation. When she reversed this process by presenting students with a mathematical formula first, it opened a gateway for a wonderful comparison between methods. I also really liked how she told students to keep their hands down when presenting the new grid because it’s “intimidating to have people think so fast”. This allows time for people who might be slower thinkers. Overall, this lesson showed me the beauty in embracing a variety of different methods and representing them in a variety of different ways.

**Ch. 3: Building Understanding of Algebraic Representation, The Border Problem, Part 2**

Cathy next discusses the importance of understanding the connections amongst algebraic representations in the form of tables, graphs, symbolic rules, and verbal rules in order to fully conceptualize functions. I think that when I demonstrate these different forms of functional representations that it will be vital to highlight their connectedness. I really enjoyed watching this lesson for a number of reasons. Cathy made careful use of the student’s visualization skills
when extending their investigation to different sized grids (6-by-6, 15-by-15, 223-by-223). This allowed students to again **visualize** a pattern but they still stayed away from a formula representation. I thought this method was particularly creative because in doing this she avoided the generic fill-in-a-value table that requires students to find a pattern instead of connecting it to a function. What she did to lead to the process of writing the function was make students create a verbal rule for the pattern they were seeing. Presenting a generic grid with color coded strips for the top and bottom as well as left and right sides allowed her to have an in-depth conversation with her students and arrive at a detailed description that would later on make understanding and formulating the function much easier. I also have the same belief as Cathy that many students don’t see connections between different representations such as a table and function with random chosen variables in it. Kids often ask, why ‘x’? What does ‘b’ mean? The letter representation confuses them and hinders them from seeing deeper mathematical properties of polynomial functions. It was also clever that she allowed each student to pick the variable they wanted, stressing that the letter only **represents** the number of squares in the border and not the border itself (which may have been the misconception had she chosen ‘b’ for all students to follow).

When students were given time to discover the formula, many struggled and were confused but I liked how Cathy said “I’m glad that you were confused” because this is not something one would normally expect a math teacher to say. By saying this, she reinforced the acceptance of making mistakes and that they actually help in making greater discoveries later on. I thought the teacher questions table on page 37 was really useful because it highlighted the variety of questions a teacher can elicit when probing for desired answers or discussions. This also shows how different questions can be applied to different situations. One that I thought was particularly important was establishing context, because diverse students don’t always have uniform
background knowledge and it’s important that their success in math isn’t hindered because of this. It was surprising to read that “exploring mathematical meanings and relationships” type questions were rarely observed in the classroom and that 95% of questions in traditional classrooms were of the “gathering information/leading students through a method” type. I want to make sure that I ask critical questions in the future that orient students to central mathematical ideas.

**Ch. 4: Defending Reasonableness, Division of Fractions**

I can see the potential danger in “learning rules without reasons” as I am not even entirely sure why the invert-and-multiply method makes sense. I thought her “convince yourself, convince a friend, convince a skeptic” was a very interesting discourse strategy because it emphasizes correct explanations rather than correct answers, which I think many students lack to master. I think that by embracing student errors, Cathy is not only able to create a comfortable learning environment among her class, but it is useful to understand these errors because they highlight misconceptions. I thought it was really insightful to see Cathy tell her students that although she was aware that some students knew the rule they were trying to get at for the day, she didn’t care. What she cared about was *making sense* of that rule which I believe that many teachers do not take the time to discuss. As I began watching this video I became very curious as to why this rule makes sense as well. Another discursive move I really want to implement in my classroom is how Cathy had all her students come to the board and present their arguments as she stayed seated at a desk near the back of the room. This norm gives students ownership over their ideas and requires them to make convincing proofs and have them challenged by other peers. Jo describes this as “all students have a role—to decide whether the representation and justification are sufficiently convincing for them” (50). In this sense, mathematics is seen as a
“collective enterprise” (51). It was great that Cathy let this student orientation take place for a while before finally presenting the traditional algorithm (instead of presenting it right away) because she allowed students time to think about the answers they came up with in a contextual sense instead of memorizing a rote procedure.

**Ch. 5: Introducing the Notion of Proof**

I believe that justification, reasoning, and proof are essential skills that students need to possess in order to truly understand mathematics. However, this can be a touchy subject, given that the word ‘prove’ alone triggers a dreadful feeling; as it had to me just years ago in my own math courses. Why do students fear the words “prove your answer” or “explain”? I think that it is because of their lack of doing so and discomfort with applying the question ‘Why?’ Students are so accustomed to reciting rules and procedures that they memorize for exams because they know it will yield the correct answer but in doing this they fail to truly understand the material. Proving certainly takes more time to both present and practice, but I feel that it is needed at the early stages of mathematics in order to generate a productive mindset when it comes to thinking mathematically. It was interesting to hear the conversation that went on about the extra credit question: prove that 2(n-1)=2n-2 because a lot of the students were having a hard time explaining why a numerical example wasn’t a sufficient proof. The students often said “I don’t know” or “I don’t know how to say it” and stuttered and struggled to formulate a fluent mathematical-based sentence. This served as a perfect example of students’ discomfort of talking about math. Even though the kids were clearly struggling in their verbal explanations, it was important that Cathy stayed silent and let them try to get their thoughts out with little question probing. I think this made students feel less pressured to give an exact answer right away. A statement I really liked in Jo’s analysis was “In essence, we do not learn what people say, we learn from what they say”
(62). This emphasized the importance of not just giving students information to retain but instead allowing them to “actively connect new knowledge with their previous conceptions and beliefs” (62).

**Ch. 6: Continuing Our Discussion of Proof, Convincing Others**

It was interesting how Cathy required everyone to think silently to themselves before discussing with their groups. I thought this was a great idea because sometimes certain group members can dominate a conversation, giving others limited time to think about and share their own thoughts. Jo described this lesson as promoting *quantitative literacy* (71) which I agree is important for engagement in the real world. I really like the quote by Elliot Eisner: “the primary aim of education is not to enable students to do well in school, but to help them do well in the lives they lead outside of school” (72). What’s missing in a lot of classrooms in order to properly do this is questioning, reasoning, and representing. All three work together to generate mathematical literacy. “Educated citizens of a quantitative society need to be able to contend, critique, analyze, and deduce—in essence, they need to reason with numbers and mathematical properties” (74). I want to make sure that I regularly incorporate these aspects into my curriculum in order to give students the tools they need to reason mathematically. Each is their own craft, however, and something as simple as questioning can have a lot more depth to it. “Questioning is both the product and process, involving an ongoing interaction between the knower, the known, and sometimes the unknown. It is a branch of knowledge itself” (72). Cathy (and I) also agree with the claim that “to ask a question is an act of cognition and literacy” (72). Hearing and writing about these methods are enlightening and endearing, but it is definitely not the same as watching Cathy’s classroom lessons unfold. Watching the students respond to other student ideas with probing questions, reason through verbal or visual
explanation, and represent their ideas in a myriad of ways showcased these idealities in a realistic manner.

**Ch. 7: Class Participation, *Through the Eyes of Students***

Getting students to communicate their thinking is a difficult task in itself. In a classroom where it is not generally common for students to vocalize their ideas, you are only left with repetitive volunteers raising their hands, and singling out a particular student who isn’t volunteering risks making them feel embarrassed, anxious, and even less likely to want to participate in the future. I agree that teachers should be experts in managing whole-class discussion, but I think this is very challenging and only mastered with experience. This is why watching Cathy perform this was particularly helpful. I liked the way Cathy phrased the importance of class participation: “You must be accountable to listening and thinking” (80) because it positively states the notion that collective student engagement is necessary for learning. The accompanying video clip for this chapter has been my favorite one yet. Although the lesson was planned for discussing the volume of a prism, Cathy quickly transforms the day’s lesson into discussing the difficulties of participation after encountering a rather painful interaction with one of the group’s reporters. I really admired the way Cathy took heart to what her students felt and how she could help them feel more comfortable about speaking in front of the class. She stressed how talking to others was a “life skill” and expressed how important she thought it was. I agree with this and I think it’s a vital skill that many teachers neglect in the mathematics classroom. One student even pointed out that Cathy was the first teacher she ever had that told her it was a good thing to be wrong. One flaw in our education system is that students associate mistakes with negativity or failure. What I want to be sure to do is to embrace misunderstandings in order to foster learning, just as Cathy had. This is well encompassed in the
quote: “mistakes must be seen by the students and the teacher as places that afford opportunities to examine errors in reasoning and thereby raise everyone’s level of analysis” (83). It was interesting to see how many students were willing to share their thoughts on this topic as opposed to a mathematically based one. What really struck me is how spontaneous this discussion was; yet it provided Cathy with valuable information for lessons to come. I would certainly love to lead a conversation with my class someday about their fears, concerns, and thoughts on why they think randomly choosing or designating a group reporter are effective and fair ways to ensure student participation and build “academic character”. After all, it is important to note that each classroom is composed of unique students who might have very different wants and needs. Through Cathy’s empathy, she established a trustful relationship with her students while simultaneously creating a sense of community among them. Both of these actions are, in my opinion, vital for student progression and enjoyment of mathematics.

**Ch. 8: Volumes of Prisms and Cylinders, Extending Prior Knowledge**

Cathy outlines the important difference between “instrumental” understanding (possession of tools such as formulas) and “relational” understanding (92). I feel that many students feel road-blocked by their instrumental understanding in that they get thrown off whenever there isn’t a clear context of how they can apply formulas they have been given. This to me signifies an inability to manipulate prior knowledge and given material to creatively solve a problem in a new way; one that Cathy would describe as a failure to master relational understanding. It’s true that we learn formulas for volume (for example) which is \( l \times w \times h \) but students rather than knowing why instead only commit this to memory. It is in this very process that the fundamental basis behind this mathematical formula is lost. Cathy’s hook was particularly effective because it got students actively engaged in recognizing rectangular and
cylindrical prisms in everyday life—particularly in a grocery store. It was interesting in her discussion when students were generating ideas about the volume formula for a cylinder. One student said the correct formula but when Cathy asked her why it makes sense she replied, “I don’t know it just seems like the right thing to do.” Discussion moved on from here but I feel like this is a crucial part in developing further understanding. The student was never told that she was correct and could’ve made an important connection to the shape and her devised formula had she been pressed further. It was enlightening, however, to see her later raise her hand anxiously and announce that he figured out why. I liked to see that Cathy did not halt the conversation once a student reported the correct answer and she says she did this to hear what others were thinking and to ensure that others had time to process what was said. “Explanations, no matter, how clear, cannot provide understanding of relationships; that important work must be done by each individual” (96). This is the main reason why the student was later able to explain why her formula worked! This discussion was extremely beneficial to all students involved because instead of simply presenting the formula, they were all able to hear different perspectives as to not only what it would be, but why. When moving onto a 3-d triangular prism, it allowed students to further apply their understanding of volume and one student even made a beautiful generalization. This not only allowed students to summarize the process for finding volume, but it portrayed the level of understanding the students had in regards to this concept to the teacher. What I really thought was valuable about Cathy’s discussion is that she was encouraging students to make connections, something that textbooks fail to regularly do: “they often isolate the different methods students need to know so that the students can practice them in a focused way. But this can also strip mathematics of the connections that are at the heart of the subject” (98-99).
Ch. 9: Surface Area, Generating Geometric Formulas

This lesson was interesting to watch because I had just previously taught a lesson on the surface area of a cylinder. I liked that she made the surface area formula derivation an investigation rather than a lecture. It was great that she also asked students to draw their incorrect predictions of what a flat cylinder would look like. This emphasized that “examining the patterns that didn’t work provided one more opportunity to reinforce the important idea that wrong answers and confusion are not only normal but essential for learning” (106). One hint I thought was particularly helpful was when Cathy asked how long the rectangle part of the flattened cylinder would need to be in order to wrap around the circle with no overlaps. This was a great way to phrase a question that still gave students the responsibility to make the discovery on their own. Something else Cathy did that I thought was great was she explained the ordering when she wrote “2\pi r” and explained that the numbers go before the variable and although \pi looks like a variable since it’s a symbol, that it’s in fact a number. She also made sure a student used correct mathematical language during an explanation (rectangle instead of box). It was notable when after the formula was presented, a student asked, “Do we have to memorize it?” Cathy addressed it very well by stressing how important it is to be able to generate formulas in the case that they forget. She said something I really liked: “the formulas come out of you, not from out of a book.” A quote I want to remember that she notes in her review by Litchtenberg says, “What you are obliged to discover yourself leaves a path in your mind which you can use again when the need arises” (110). I also liked how Cathy wrote in her review: “I always try hard to listen to what students are really saying rather than what I want to hear” (109). I think this is so important because as teachers we have to understand that our students are going to have vastly different perspectives than ours, and the way they phrase their thoughts might not be the same way that we phrase ours. Another one of her favorite sayings
that coincides with this thought is: “Obvious’ is the most dangerous word in mathematics” (109). What might appear “easy” or obvious to an experienced mathematical expert might be completely foreign or hidden to outsiders. An important issue Jo highlights is that many students perceive math to be a one-way path and they enjoy it because “it is or it isn’t” and this is identical to the way I perceived math coming into college. I remember writing in my college entry essay how I highly enjoyed that math always has one answer. After spending four years and the University and especially reading this book, it is evident that this is certainly not the case. As a future teacher I want to be sure to emphasize to my students that they should not take on a role of “received knowers” but instead free analyzers that can arrive at a solution in a number of different ways.

**Summative Piece**

Reading this book and watching the videos allowed me to learn from a professional standpoint about the practice of teaching. The deliberate pedagogical moves Cathy employed coupled with her emphasis on justification, representation, sociomathematical norms, and connectedness between algebraic and geometric ideas encompassed the complexity and intricacy of the teaching profession. As Cathy supports both small-group and whole-class discussions, she orchestrates a connection between the student and the content, between the student and herself, as well as (and most notably) between student to student. “Cases of teaching provide a special opportunity for teachers to learn and grow, not by communicating answers or presenting model teaching, but by prompting questions” (3). Reading this definitely urged me to question my own teaching practices and imagine how I might employ some of these techniques into my own student teaching experience next spring. Though I may have been offered “general strategies and principles about education, such abstract knowledge can be extremely powerful, but it leaves
to teachers the task of translating into practical action in their own classroom” (4). Cathy’s videos were extremely resourceful observations in which I was able to see these idealistic theories incorporated into a real classroom. I enjoyed hearing Cathy’s past students speak of their appreciation for the multiple forms of representations and explanations that constantly took place in her class because it allowed “everybody to get it.” It was also interesting how much they loved when Cathy said it was too quiet and that they needed to talk. Although this could be taken the wrong way, it shows how different her teaching style was from ones they experienced in the past. They also spoke of how meaningful it was that Cathy appreciated their mistakes because it didn’t make them feel bad about themselves and they were even able to see how it helped foster learning. It was surprising to hear a student in the interview say they didn’t even know what explaining was or how to use it prior to entering Cathy’s class. I believe many students are afraid of verbalizing their mathematical thoughts and Cathy’s videos provided excellent examples of how to conquer this. The beginning of this chapter mentions pedagogical content knowledge, described as “the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction” (5). The source from which teachers can learn “such rich and specialized knowledge” is significantly limited. Cathy’s book offered a wonderful combination of content and pedagogy that helped me to further develop my stance on what pedagogical content knowledge really is and how it is supposed to look in an everyday classroom. On page 88 of the text, Jo makes note of practices that “maintain respect for the ‘integrity of mathematics’” performed by successful teachers that I would like to briefly summarize:

- Strategically call on particular students who might offer helpful questions or representations
Ask students to hold on to a thought if it changes discussion direction
Ask a student who produced interesting work to prepare to show others
Stress the importance of particular students’ comments using tone and emphasis
Add ideas, questions, and linkages between different student ideas

Reviewing this list, I can identify specific lesson points when Cathy has implemented these strategies. Her experience and wisdom has provided me with useful tools and specific discursive moves I can perform in my future classroom in order to promote equity and ensure student success for all.

Works Cited

Jo Boaler and Cathy Humphreys, *Connecting mathematical ideas: Middle School Video Cases to Support Teaching and Learning*, Heinemann, Portsmouth NH, 2005